Statistical properties of a class of nonlinear systems driven by colored multiplicative Gaussian noise

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We derive the time-dependent univariate and bivariate probability distribution function for an overdamped system with a quadratic potential driven by colored Gaussian noise, whose amplitude depends on the system state x as $|x|^{\alpha}$. Particular attention is paid to the effect of the correlation function of the noise on the statistical properties of the system. We obtain exact expressions for the fractional moments as well as the correlation function of the system and calculate the fractal dimension. We also consider the special case of a constant potential and determine the criteria for anomalous diffusion and stochastic localization of free particles.

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I. INTRODUCTION

Models with state-dependent (multiplicative) noise find numerous applications in many different fields of science, for example, in quantum optics [1], biology [2–4], noise-induced transitions [5,6], growth phenomena [7], reaction-diffusion models of chemical systems and epidemics [8–11], and economic activities [12]. They are also currently studied as simple models that generate power-law probability distribution functions (PDFs) [13]. A number of natural, social, and economic phenomena are claimed to be described by power-law distributions, and such distributions are considered the signature of complex self-organizing systems [14,15].

Systems with state-dependent noise are usually modeled by a discrete- or continuous-time version of the multiplicative Langevin equation, i.e., a Langevin equation in which the noise $f(t)$ is multiplied by a function of the system state $x(t)$. Since the Langevin equation relates the state of the system $x(t)$ to the noise $f(t)$, one expects that the statistical characteristics of $x(t)$ can be expressed in terms of the given statistical characteristics of $f(t)$. An explicit solution, however, cannot always be found, even for the linear multiplicative Langevin equation, a generic model for generating power-law PDFs [13]. For the continuous-time version of the nonlinear multiplicative Langevin equation, no general method exists to determine the PDFs of the system in terms of the noise for arbitrary $f(t)$. The problem simplifies significantly, if $f(t)$ is Gaussian white noise. Then $x(t)$ is a Markovian diffusion process [16], and its univariate PDF and transition probability density satisfy the Fokker-Planck equation, which can be solved exactly in specific cases [5,17,18].

However, various physical effects are induced only by colored noise, which has a nonzero correlation time, and in these cases the white noise approximation represents an oversimplification. Ratchet systems are one example; here a nonzero current of Brownian particles results from the perturbation of an asymmetric periodic potential by external correlated random or periodic forces [19]. Linear systems with additive colored noise are another example. In contrast to white noise, colored noise can give rise to anomalous diffusion of free particles without dissipation [20], with nonlocal dissipation [21], with time-dependent friction [22], and can lead to anomalous diffusion and stochastic localization of damped classical [23] and quantum [24] particles. Systematic studies of the statistical properties of nonlinear systems driven by colored noise have barely begun to be undertaken.

In this paper, we study in detail nonlinear systems whose state $x(t)$ evolves according to the multiplicative Langevin equation

$$\dot{x}(t)+\kappa x(t)=|x(t)|^{\alpha}f(t)\ [x(0)=x_0>0], \tag{1.1}$$

where $\kappa\geqslant 0$, α is a real-valued parameter, and $f(t)$ is a noise with zero mean and known statistical characteristics. Equation (1.1) describes a wide class of random processes. Specifically, if $f(t)$ is Gaussian white noise, then $x(t)$ is the Wiener process if $\kappa=0$ and $\alpha=0$, the Ornstein-Uhlenbeck process if $\kappa>0$ and $\alpha=0$ [25], and the lognormal process if $\kappa=0$ and $\alpha=1$ [26]. Further, multiplicative noise with $\alpha=1/2$ occurs in models of lasers [1], and in models of chemical reactions and epidemics [8–11]. The latter belong to the universality class that can be represented by the Langevin equation of Reggeon field theory. The spatially homogeneous version of that equation coincides with Eq. (1.1) for small x.

An interesting feature of Eq. (1.1) is the fact that for $0<\alpha<1$ the solution is not unique at $x=0$; there are two solutions that pass through zero. Physical considerations determine the appropriate choice for each model or application. If the point $x=0$ should be considered to be an absorbing point, as for example in the chemical and epidemic models mentioned above, then the solution of Eq. (1.1) coincides with the solution of the equation

$$[\dot{x}(t)+\kappa x(t)]|x(t)|^{-\alpha}=f(t)\ [x(0)=x_0], \tag{1.2}$$

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for $0 \leqslant t < t_{fp}$ (t_{fp} is the random first-passage time from $x(0)=x_0$ to $x(t_{fp})=0$) and $x(t)\equiv 0$ for $t \geqslant t_{fp}$. This case will be considered elsewhere [27]. In this paper, we study the case where $x(t)$ represents the (generalized) coordinate of an overdamped particle moving in a parabolic potential. The point $x=0$ should then be considered to be a regular point, and the solution of Eq. (1.1) coincides with the solution of Eq. (1.2) for all times $t \geqslant 0$.

Our central result is an analytical expression for the single time and two-time probability density function of the random process $x(t)$ governed by the Langevin equation (1.2). The temporal evolution of $x(t)$ is determined by the competing effects of the systematic restoring force $-\kappa x(t)$ and the random driving force $|x(t)|^{\alpha} f(t)$. The effects of this competition are studied by calculating the fractional moments of the single-time or univariate PDF $P_x(x,t)$ and exploring their short- and long-time behavior. For the case $\kappa=0$, these moments are useful to characterize the diffusive behavior of free particles. We find that, depending on the noise intensity and the exponent α, colored multiplicative Gaussian noise can lead to stochastic localization, normal diffusion, subdiffusion, and superdiffusion. An analysis of the temporal evolution of $P_x(x,t)$ provides further insight into the competition between the systematic and random force. We find that the opposing effects of these two forces lead to temporal bimodality. As far as numerical characteristics of the two-time or bivariate PDF are concerned, we derive expressions for the coefficient of correlation and show that correlations between $x(t)$ and $x(t_1)$ persist as $|t-t_1|\to\infty$ only for free particles in the case of stochastic localization. To characterize the irregularities of the sample paths of the random process, we calculate their fractal dimension. Only colored Gaussian noise whose correlation function diverges as a power law at zero leads to fractal sample paths.

The paper is structured as follows. In Sec. II, we solve Eq. (1.2) for the general case of arbitrary noise $f(t)$ and exclude values of the parameter α for which the system reaches infinity with nonzero probability on any finite time interval. In Sec. III, we derive the uni- and bivariate PDFs of $x(t)$ for stationary Gaussian noise $f(t)$. In Sec. IV, we obtain exact expressions for the fractional moments of $x(t)$ and their short- and long-time asymptotics. In the same section we determine the criteria for anomalous diffusion and stochastic localization of free particles. In Sec. V, we study the time evolution of the univariate PDF analytically and numerically. In Sec. VI, we calculate the correlation function and the coefficient of correlation, and in Sec. VII we obtain the fractal dimension of $x(t)$. We summarize our results in Sec. VIII.

II. SOLUTION OF THE LANGEVIN EQUATION

Our aim is to express the statistical properties of $x(t)$ in terms of the given statistical characteristics of the random driving force $f(t)$. To this end, we need to obtain an explicit solution of the Langevin equation (1.2). We introduce the new variable $y(t)=x(t)\exp(\kappa t)$ and reduce the equation to

$$\dot{y}(t)|y(t)|^{-\alpha}=e^{\omega t}f(t), \tag{2.1}$$

where $\omega=(1-\alpha)\kappa$. Taking into account that $y(0)=x_0$, we obtain

$$\int_{x_0}^{y(t)}\frac{dy'}{|y'|^{\alpha}}=\int_0^t dt' e^{\omega t'} f(t'). \tag{2.2}$$

If $\alpha\neq 1$, then Eq. (2.2) yields

$$y(t)|y(t)|^{-\alpha}=x_0^{1-\alpha}+(1-\alpha)\int_0^t dt' e^{\omega t'} f(t'), \tag{2.3}$$

and the solution of Eq. (1.2) is given by

$$x(t)=[x_0^{1-\alpha}e^{-\omega t}+q(t)]|x_0^{1-\alpha}e^{-\omega t}+q(t)|^{\alpha/(1-\alpha)}, \tag{2.4}$$

where

$$q(t)=(1-\alpha)\int_0^t dt' e^{-\omega(t-t')}f(t'). \tag{2.5}$$

For $\alpha>1$, Eq. (2.4) leads to $|x(t)|\to\infty$ as $x_0^{1-\alpha}e^{-\omega t}+q(t)\to 0$. If the noise $f(t)$ has an infinite range of values, then the random function $q(t)$ has the same range. In this case, the probability that the equation $x_0^{1-\alpha}e^{-\omega t}+q(t)=0$ has at least one solution on any interval $(0,t)$ is nonzero. This implies that the state of the system $x(t)$ reaches infinity on any finite time interval with nonzero probability. Further, for $x_0^{1-\alpha}e^{-\omega t}+q(t)=+0$ and $x_0^{1-\alpha}e^{-\omega t}+q(t)=-0$, i.e., for an infinitesimally small change of time, Eq. (2.4) yields $x(t)=+\infty$ and $x(t)=-\infty$, respectively. To exclude this unphysical behavior, we will only consider the case $\alpha<1$.

If $\alpha=1$, then the integral on the left-hand side of Eq. (2.2) goes to $-\infty$ as $y(t)\to 0$ and to $+\infty$ as $y(t)\to+\infty$, i.e., $y(t)\geqslant 0$ for all times. In this case the solution of Eq. (2.2) has the form $y(t)=x_0 \exp w(t)$, where

$$w(t)=\int_0^t dt' f(t'), \tag{2.6}$$

and

$$x(t)=x_0 \exp[-\kappa t+w(t)]. \tag{2.7}$$

Note that for $\alpha=1$, Eqs. (1.2) and (1.1) are equivalent.

III. BIVARIATE AND UNIVARIATE PDF

We have obtained an explicit expression for $x(t)$ in terms of a functional of $f(t)$, namely, $w(t)$ for $\alpha=1$ and $q(t)$ for $\alpha<1$, respectively. These functionals represent the cumulative effect of the random driving force from the initial instant up to time t. For $\alpha=1$, $w(t)$ is simply the integral over $f(t)$, whereas for $\alpha<1$, the past influence of the driving force is weighted by an exponential kernel. The time-dependent univariate and bivariate PDF of the multiplicative noise system (1.1) can now be determined if the bivariate PDF of the force functionals $q(t)$ and $w(t)$, respectively, can be obtained. This is certainly the case for Gaussian noise as explained below.

A. Bivariate PDF

Let $P_x(x,t;x_1,t_1)$ and $P_q(q,t;q_1,t_1)$ be the bivariate PDFs that $x(t)=x$ and $x(t_1)=x_1$, and $q(t)=q$ and $q(t_1)=q_1$, respectively. According to Eq. (2.4) the relation

$$q(t)=x(t)|x(t)|^{-\alpha}-x_0^{1-\alpha}e^{-\omega t}, \tag{3.1}$$

holds, and a one-to-one correspondence exists between $x(t)$ and $q(t)$. This implies that $P_x(x,t;x_1,t_1)|dx\,dx_1| = P_q(q,t;q_1,t_1)|dq\,dq_1|$, and

$$P_x(x,t;x_1,t_1)=P_q(q,t;q_1,t_1)\left|\frac{\partial(q,q_1)}{\partial(x,x_1)}\right|, \tag{3.2}$$

where

$$\left|\frac{\partial(q,q_1)}{\partial(x,x_1)}\right|=\frac{(1-\alpha)^2}{|xx_1|^{\alpha}}, \tag{3.3}$$

is the Jacobian. If the bivariate PDF $P_q(q,t;q_1,t_1)$ is known, then the bivariate PDF $P_x(x,t;x_1,t_1)$ for $\alpha<1$ is given by

$$P_x(x,t;x_1,t_1)=\frac{(1-\alpha)^2}{|xx_1|^{\alpha}}P_q(x|x|^{-\alpha}-x_0^{1-\alpha}e^{-\omega t},t;x_1|x_1|^{-\alpha} -x_0^{1-\alpha}e^{-\omega t_1},t_1). \tag{3.4}$$

Using the relation

$$w(t)=\ln\frac{x(t)}{x_0}+\kappa t, \tag{3.5}$$

which follows from Eq. (2.7), we obtain in the same way for $\alpha=1$,

$$P_x(x,t;x_1,t_1)=\frac{1}{x\,x_1}P_w\left(\ln\frac{x}{x_0}+\kappa t,t;\ln\frac{x_1}{x_0}+\kappa t_1,t_1\right), \tag{3.6}$$

$(x,x_1\geqslant 0)$, where $P_w(w,t;w_1,t_1)$ is the bivariate PDF that $w(t)=w$ and $w(t_1)=w_1$.

The bivariate PDFs of $q(t)$ and $w(t)$ are easily obtained for the case of a Gaussian random force. Since these functionals depend linearly on $f(t)$, see Eqs. (2.5) and (2.6), they are themselves Gaussian processes. As is well known, a Gaussian process is fully defined by its mean value and its correlation function. In our case $\langle f(t)\rangle=0$, and therefore $\langle q(t)\rangle=0$ and $\langle w(t)\rangle=0$, where $\langle\rangle$ denotes averaging with respect to the noise $f(t)$. To fully determine the above bivariate PDFs, we need to express the correlation functions of $q(t)$, $\langle q(t)q(t')\rangle\equiv R_q(t,t')$, and of $w(t)$, $\langle w(t)w(t')\rangle \equiv R_w(t,t')$, in terms of the correlation function $\langle f(t)f(t')\rangle \equiv R(|t-t'|)$ of the stationary Gaussian noise $f(t)$. From Eqs. (2.5) and (2.6) we obtain

$$R_q(t,t')=(1-\alpha)^2e^{-\omega(t+t')}\int_0^t d\tau\int_0^{t'}d\tau'\,e^{\omega(\tau+\tau')} \times R(|\tau-\tau'|), \tag{3.7}$$

and

$$R_w(t,t')=\int_0^t d\tau\int_0^{t'}d\tau' R(|\tau-\tau'|). \tag{3.8}$$

Introducing the new variables $u=\tau-\tau'$, $v=\tau+\tau'$ and defining

$$F_\omega(z)=\frac{1}{\omega}\int_0^z du R(u)\sinh[\omega(z-u)], \tag{3.9}$$

we can reduce Eqs. (3.7) and (3.8) after some algebra to

$$R_q(t,t')=(1-\alpha)^2[e^{-\omega t'}F_\omega(t)+e^{-\omega t}F_\omega(t')-F_\omega(t-t')], \tag{3.10}$$

and

$$R_w(t,t')=F_0(t)+F_0(t')-F_0(t-t'), \tag{3.11}$$

where $F_0(t)=\lim_{\omega\to 0}F_\omega(t)$. Note that $F_\omega(-z)=F_\omega(z)$, since $R(-u)=R(u)$. Further, we have that

$$e^{-\omega t}F_\omega(t)=\frac{1}{2}\left\langle\left(\int_0^t d\tau e^{-\omega(t-\tau)}f(\tau)\right)^2\right\rangle, \tag{3.12}$$

which implies that $F_\omega(t)\geqslant 0$ and $F_\omega(t)=0$ only for $t=0$.

Using the well-known expression for the bivariate PDF of a Gaussian process [28], we obtain from Eqs. (3.4) and (3.6)

$$\begin{aligned}P_x(x,t;x_1,t_1)=&\frac{(1-\alpha)^2|xx_1|^{-\alpha}}{2\pi\sigma_q(t)\sigma_q(t_1)\sqrt{1-r_q^2(t,t_1)}}\\ &\times\exp\Bigg\{-\frac{1}{2[1-r_q^2(t,t_1)]}\\ &\times\Bigg[\frac{1}{\sigma_q^2(t)}\left(\frac{x}{|x|^{\alpha}}-x_0^{1-\alpha}e^{-\omega t}\right)^2\\ &+\frac{1}{\sigma_q^2(t_1)}\left(\frac{x_1}{|x_1|^{\alpha}}-x_0^{1-\alpha}e^{-\omega t_1}\right)^2\\ &-\frac{2r_q(t,t_1)}{\sigma_q(t)\sigma_q(t_1)}\left(\frac{x}{|x|^{\alpha}}-x_0^{1-\alpha}e^{-\omega t}\right)\\ &\times\left(\frac{x_1}{|x_1|^{\alpha}}-x_0^{1-\alpha}e^{-\omega t_1}\right)\Bigg]\Bigg\},\end{aligned} \tag{3.13}$$

for $\alpha<1$, and

$$P_x(x,t;x_1,t_1)=\frac{(xx_1)^{-1}}{2\pi\sigma_w(t)\sigma_w(t_1)\sqrt{1-r_w^2(t,t_1)}}\times\exp\left\{-\frac{1}{2[1-r_w^2(t,t_1)]}\times\left[\frac{1}{\sigma_w^2(t)}\left(\ln\frac{x}{x_0}+\kappa t\right)^2+\frac{1}{\sigma_w^2(t_1)}\times\left(\ln\frac{x_1}{x_0}+\kappa t_1\right)^2-\frac{2r_w(t,t_1)}{\sigma_w(t)\sigma_w(t_1)}\times\left(\ln\frac{x}{x_0}+\kappa t\right)\left(\ln\frac{x_1}{x_0}+\kappa t_1\right)\right]\right\}, \quad (3.14)$$

for $\alpha=1$. Here

$$\sigma_q^2(t)\equiv R_q(t,t)=2(1-\alpha)^2\frac{e^{-\omega t}}{\omega}\int_0^t duR(u)\sinh[\omega(t-u)], \quad (3.15)$$

is the dispersion of $q(t)$,

$$\sigma_w^2(t)\equiv R_w(t,t)=2\int_0^t duR(u)(t-u), \quad (3.16)$$

is the dispersion of $w(t)$, and

$$r_{q,w}(t,t_1)=\frac{R_{q,w}(t,t_1)}{\sigma_{q,w}(t)\sigma_{q,w}(t_1)}, \quad (3.17)$$

are the coefficients of correlation, which satisfy the condition $|r_{q,w}(t,t_1)|\leqslant 1$ [29].

B. Univariate PDF

To obtain the univariate PDF $P_x(x,t)$ we can proceed in the same way as for the bivariate PDF, or we can simply eliminate one variable by integration,

$$P_x(x,t)=\int_{-\infty}^{\infty}dx_1P_x(x,t;x_1,t_1). \quad (3.18)$$

Substituting expression (3.4) into Eq. (3.18), using the transformation of variables $y=x_1|x_1|^{-\alpha}$, and taking into account that integration of $P_q(q,t;q_1,t_1)$ over q_1 yields the univariate PDF $P_q(q,t)$, we find for $\alpha<1$,

$$P_x(x,t)=\frac{1-\alpha}{|x|^\alpha}P_q\left(\frac{x}{|x|^\alpha}-x_0^{1-\alpha}e^{-\omega t},t\right). \quad (3.19)$$

In the same way we find for $\alpha=1$,

$$P_x(x,t)=\frac{1}{x}P_w\left(\ln\frac{x}{x_0}+\kappa t,t\right). \quad (3.20)$$

If $f(t)$ is a Gaussian noise, Eqs. (3.19) and (3.20) yield

$$P_x(x,t)=\frac{1-\alpha}{\sqrt{2\pi}\sigma_q(t)|x|^\alpha}\times\exp\left\{-\frac{1}{2\sigma_q^2(t)}\left(\frac{x}{|x|^\alpha}-x_0^{1-\alpha}e^{-\omega t}\right)^2\right\}, \quad (3.21)$$

for $\alpha<1$, and

$$P_x(x,t)=\frac{1}{\sqrt{2\pi}\sigma_w(t)x}\exp\left\{-\frac{1}{2\sigma_w^2(t)}\left(\ln\frac{x}{x_0}+\kappa t\right)^2\right\} \quad (3.22)$$

for $\alpha=1$. Expressions (3.14) and (3.22) are the bivariate and univariate PDFs of the logarithmic-normal (lognormal) distribution [26]. By analogy, we call the probability distribution, whose bivariate and univariate PDFs are given by Eqs. (3.13) and (3.21), the power-normal distribution.

It is not difficult to verify that the univariate PDFs (3.21) and (3.22) satisfy the Fokker-Planck equation

$$\frac{\partial}{\partial t}P_x(x,t)=\frac{\partial}{\partial x}[\kappa x-\Delta_\omega(t)\alpha x|x|^{2(\alpha-1)}]P_x(x,t)+\Delta_\omega(t)\frac{\partial^2}{\partial x^2}|x|^{2\alpha}P_x(x,t), \quad (3.23)$$

where the function

$$\Delta_\omega(t)=\int_0^t duR(u)e^{-\omega u}=\begin{cases}\dfrac{\sigma_q(t)\dot{\sigma}_q(t)+\omega\sigma_q^2(t)}{(1-\alpha)^2}, & \alpha<1,\\ \sigma_w(t)\dot{\sigma}_w(t), & \alpha=1,\end{cases} \quad (3.24)$$

is the exponentially weighted time-dependent intensity of $f(t)$. Specifically, if $f(t)$ is Gaussian white noise, then $R(u)=2\Delta\delta(u)$ [Δ is the white noise intensity, $\delta(u)$ is the δ function] and Eq. (3.24) yields $\Delta_\omega(t)=\Delta$. In that case, $x(t)$ is a Markovian diffusion process, and Eq. (3.23) corresponds to the Stratonovich interpretation [30] of Eq. (1.1). We emphasize that for colored noise $f(t)$ the random process $x(t)$ is not Markovian, in spite of the fact that $P_x(x,t)$ obeys a Fokker-Planck equation. (For a Markovian process, it is the transition probability density, and not only the univariate PDF, that obeys a Fokker-Planck equation.) We will exploit the fact that the univariate PDF (3.21) obeys a Fokker-Planck equation in another paper [27] to obtain the statistical properties of $x(t)$ with an absorbing boundary at $x=0$.

IV. FRACTIONAL MOMENTS

In the previous section, we have achieved the main goal of this work, namely, to express the statistical properties of the state variable $x(t)$ in terms of the statistical characteristics of the driving force $f(t)$ for the case of colored Gaussian noise. Though Eqs. (3.13), (3.14), (3.21), and (3.22) provide explicit expressions for the bivariate and univariate PDFs, it is helpful for our understanding of colored noise systems of type (1.1) to consider also a more concise description and determine numerical characteristics of the random process $x(t)$. Moments are of particular interest for applications, and we begin our analysis of the temporal evolution of Eq. (1.1) by calculating the time-dependent fractional moments of $x(t)$. They are defined as follows:

$$m_r^{\nu}(t)=\int_{-\infty}^{\infty}dxP_x(x,t)|x|^{r-\nu}x^{\nu}, \qquad (4.1)$$

where r is a real number, and $\nu=0$ or 1. In the case of the power-normal distribution, the fractional moments $m_r^1(t)$ characterize its asymmetry, and always $m_r^0(t)\geqslant m_r^1(t)$. For the lognormal distribution $P_x(x,t)\equiv 0$ if $x<0$, and $m_r^0(t)=m_r^1(t)\equiv m_r(t)$. Fractional moments with $r>0$ are a useful tool to characterize the behavior of $P_x(x,t)$ as $|x|\to\infty$, and those with $r<0$ provide information about the behavior of $P_x(x,t)$ in the vicinity of $x=0$. The convergence or divergence of $m_r^{\nu}(t)$ for a particular real r allows us to draw conclusions about the functional behavior of the univariate PDF as $|x|\to\infty$ and $x\to 0$, respectively.

First we calculate the fractional moments for $\alpha<1$, i.e., for the case of the power-normal distribution. Writing Eq. (4.1) as

$$m_r^{\nu}(t)=\int_0^{\infty}dx[P_x(x,t)+(-1)^{\nu}P_x(-x,t)]x^r, \qquad (4.2)$$

and using Eq. (3.21), we obtain

$$m_r^{\nu}(t)=\frac{\sigma_q^{\xi-1}(t)}{\sqrt{2\pi}}a^{\xi}(t)\int_0^{\infty}dv v^{\xi-1}[e^{-a^2(t)(v-1)^2/2}+(-1)^{\nu}e^{-a^2(t)(v+1)^2/2}], \qquad (4.3)$$

where $\xi=1+r/(1-\alpha)$, and $a(t)=x_0^{1-\alpha}e^{-\omega t}/\sigma_q(t)$. According to Eq. (4.3), all fractional moments diverge, if $\xi\leqslant 0$, that is, if $r\leqslant\alpha-1$. For $\xi>0$, we use the integral representation of the Weber parabolic cylinder functions [31]

$$D_{-\mu}(z)=\frac{e^{-z^2/4}}{\Gamma(\mu)}\int_0^{\infty}dyy^{\mu-1}e^{-y^2/2-zy}\ (\mu>0), \qquad (4.4)$$

[$\Gamma(\mu)=\int_0^{\infty}dyy^{\mu-1}e^{-y}$ is the gamma function], and reduce Eq. (4.3) to

$$m_r^{\nu}(t)=\frac{\Gamma(\xi)}{\sqrt{2\pi}}e^{-a^2(t)/4}\sigma_q^{\xi-1}(t)\{D_{-\xi}[-a(t)]+(-1)^{\nu}D_{-\xi}[a(t)]\}. \qquad (4.5)$$

If $\xi=n+1$ ($n=0,1,\ldots$), then [32]

$$D_{-n-1}(z)=\sqrt{\frac{\pi}{2}}\frac{(-1)^n}{n!}e^{-z^2/4}\frac{d^n}{dz^n}\left(e^{z^2/2}\ \mathrm{erfc}\ \frac{z}{\sqrt{2}}\right), \qquad (4.6)$$

where $\mathrm{erfc}\,(z)=(2/\sqrt{\pi})\int_z^{\infty}dt\exp(-t^2)$ is the complementary error function. Specifically, for $n=0$ we have

$$D_{-1}(z)=\sqrt{\frac{\pi}{2}}e^{z^2/4}\ \mathrm{erfc}\ \frac{z}{\sqrt{2}}, \qquad (4.7)$$

and, since $\mathrm{erfc}\,(z)+\mathrm{erfc}\,(-z)=2$, Eq. (4.5) yields $m_0^0(t)=1$, i.e., $P_x(x,t)$ is properly normalized.

For the lognormal distribution, i.e., for $\alpha=1$, Eq. (4.1) yields

$$m_r(t)=\frac{1}{\sqrt{2\pi}\sigma_w(t)}\int_0^{\infty}dxx^{r-1}\exp\left[-\frac{1}{2\sigma_w^2(t)}\times\left(\ln\frac{x}{x_0}+\kappa t\right)^2\right]. \qquad (4.8)$$

To evaluate the integral, we introduce the new variable $y=\ln(x/x_0)+\kappa t$, and find

$$m_r(t)=x_0^r\exp\left(\frac{1}{2}r^2\sigma_w^2(t)-r\kappa t\right), \qquad (4.9)$$

which is valid for all r. In particular, $m_0(t)=1$, i.e., $P_x(x,t)$ is properly normalized.

Moments provide a concise means of characterizing the time evolution of a random process. To gain insight into the motion of a particle in a quadratic potential driven by multiplicative colored Gaussian noise, we evaluate the asymptotic behavior of the fractional moments for $t\to 0$ and $t\to\infty$.

A. Short-time behavior

First we determine the asymptotic behavior of $m_r^{\nu}(t)$ and $m_r(t)$ for $t\to 0$. We consider the case that the leading asymptotic term of the correlation function of the noise $R(u)$ obeys a power law, i.e., $R(u)\sim c_{\alpha}u^{-\beta}$ as $u\to 0$. Here c_{α} is a positive parameter, which has the dimension of $x_0^{2(1-\alpha)}t^{\beta-2}$, and $0\leqslant\beta<1$. [The inequality $\beta\geqslant 0$ follows from the condition $R(0)\geqslant R(u)$, which is valid for arbitrary stationary process $f(t)$, and the inequality $\beta<1$ from the condition that the integral in Eq. (3.15) converges at the lower limit.] In this case, Eqs. (3.15) and (3.16) yield

$$\begin{pmatrix}\sigma_q^2(t)\\ \sigma_w^2(t)\end{pmatrix}\sim\begin{pmatrix}c_{\alpha}(1-\alpha)^2\\ c_1\end{pmatrix}\frac{2t^{2-\beta}}{(1-\beta)(2-\beta)}, \qquad (4.10)$$

as $t\to 0$. Since $a(t)\to\infty$ if $t\to 0$, we use the Laplace method [33] to obtain the following asymptotic formulas for the integrals in Eq. (4.3):

$$\int_0^{\infty}dvv^{\xi-1}e^{-a^2(t)(v-1)^2/2}\sim\frac{\sqrt{2\pi}}{a(t)}\left[1+\frac{(1-\xi)(2-\xi)}{2a^2(t)}\right],$$

$$\int_0^{\infty}dvv^{\xi-1}e^{-a^2(t)(v+1)^2/2}\sim\Gamma(\xi)\frac{e^{-a^2(t)/2}}{a^{2\xi}(t)}, \qquad (4.11)$$

as $a(t)\to\infty$. For $\kappa>0$, Eqs. (4.3), (4.9), and (4.11) lead to the same asymptotic formula for $m_r^{\nu}(t)$ and $m_r(t)$,

$$\begin{pmatrix} m_r^{\nu}(t) \\ m_r(t) \end{pmatrix} \sim x_0^r (1-r\kappa t) \ (t\to 0), \tag{4.12}$$

and for $\kappa=0$, they yield

$$m_r^{\nu}(t) \sim x_0^r \left(1 + \frac{r(r+\alpha-1)c_{\alpha}}{(1-\beta)(2-\beta)x_0^{2(1-\alpha)}} t^{2-\beta} \right), \tag{4.13}$$

and

$$m_r(t) \sim x_0^r \left(1 + \frac{r^2 c_1}{(1-\beta)(2-\beta)} t^{2-\beta} \right), \tag{4.14}$$

$(t\to 0)$. Here $r\in(\alpha-1,\infty)$ and $r\in(-\infty,\infty)$ if $\alpha<1$ and $\alpha=1$, respectively. Note that the fractional moments $m_r^{\nu}(t)$ do not depend on ν, since for $t\to 0$ the support of the univariate PDF $P_x(x,t)$ is a small vicinity of the point $x=x_0$.

Our results show the expected behavior. For $\kappa>0$, fractional moments with positive r decrease and those with negative r increase with time. This behavior indicates that the short-time evolution of the particle is dominated by the systematic force that drives the particle towards the origin. For a flat potential, $\kappa=0$, i.e., a free particle, the short-time motion is of course driven by the random force. The broadening of the PDF due to the multiplicative noise is reflected by the fact that the moments that probe the behavior near zero, i.e., $r<0$, as well as those that probe the behavior for large $|x|$, i.e., $r>0$, increase with time.

B. Long-time behavior

To address the long-time behavior of the particle, $t\to\infty$, we need to consider four cases separately, namely, $\alpha<1$ and $\kappa>0$, $\alpha<1$ and $\kappa=0$, $\alpha=1$ and $\kappa>0$, and $\alpha=1$ and $\kappa=0$.

1. $\alpha<1$, $\kappa>0$

In this case $a(\infty)=0$, and Eq. (3.15) yields

$$\sigma_q^2(\infty) = (1-\alpha)^2 \frac{1}{\omega} \int_0^{\infty} du R(u) e^{-\omega u}. \tag{4.15}$$

[Since $R(u)\to 0$ as $u\to\infty$, $\sigma_q^2(\infty)<\infty$.] Using the formula

$$D_{-\xi}(0) = 2^{\xi/2-1} \frac{\Gamma(\xi/2)}{\Gamma(\xi)}, \tag{4.16}$$

which follows from Eq. (4.4), we obtain

$$m_r^{\nu}(\infty) = \frac{\Gamma(\xi/2)}{\sqrt{2\pi}} 2^{\xi/2} \sigma_q^{\xi-1}(\infty) \frac{1+(-1)^{\nu}}{2}, \tag{4.17}$$

$(\xi>0)$. In this case all fractional moments with $r>\alpha-1$ have a finite value, and according to Eq. (3.21) the stationary PDF $P_{st}(x)=P_x(x,\infty)$ has the form

$$P_{st}(x) = \frac{1-\alpha}{\sqrt{2\pi}\sigma_q(\infty)|x|^{\alpha}} \exp\left(-\frac{|x|^{2(1-\alpha)}}{2\sigma_q^2(\infty)} \right). \tag{4.18}$$

Note that $P_{st}(x)$ is even, as is also reflected by $m_r^1(\infty)=0$.

As expected, our results show that the interplay between the systematic restoring force and the random driving force achieves a balance in the long term and results in a stationary PDF.

2. $\alpha<1$, $\kappa=0$

This is the case of a constant potential, i.e., the case of a free particle. As mentioned in the Introduction, free particles described by Langevin equations with additive colored noise can display anomalous diffusion. Here we investigate the effect of multiplicative colored noise on the diffusive behavior of free particles. For $\alpha<1$ and $\kappa=0$, Eq. (3.15) is reduced to

$$\sigma_q^2(t) = 2(1-\alpha)^2 \int_0^t du R(u)(t-u). \tag{4.19}$$

According to Ref. [23], if

$$\dot{F}_0(t) = \int_0^t du R(u) = o(1/t) \ (t\to\infty), \tag{4.20}$$

then $\sigma_q^2(\infty)<\infty$, and if $0<R\leqslant\infty$, where $R=\int_0^{\infty} du R(u)$ is the noise intensity, or if $R=0$, but Eq. (4.20) does not hold, then $\sigma_q^2(\infty)=\infty$. This implies that all fractional moments are finite if Eq. (4.20) is fulfilled,

$$m_r^{\nu}(\infty) = \frac{\Gamma(\xi)}{\sqrt{2\pi}} e^{-a^2(\infty)/4} \sigma_q^{\xi-1}(\infty) \{ D_{-\xi}[-a(\infty)] + (-1)^{\nu} D_{-\xi}[a(\infty)] \}, \tag{4.21}$$

with $a(\infty)=x_0^{1-\alpha}/\sigma_q(\infty)$, and the stationary PDF is given by

$$P_{st}(x) = \frac{1-\alpha}{\sqrt{2\pi}\sigma_q(\infty)|x|^{\alpha}} \exp\left(-\frac{(x|x|^{-\alpha}-x_0^{1-\alpha})^2}{2\sigma_q^2(\infty)} \right). \tag{4.22}$$

In contrast to the previous case, $P_{st}(x)$ is not even. Indeed, Eq. (4.22) shows that $P_{st}(-x)\neq P_{st}(x)$.

These results show that a free particle driven by multiplicative colored Gaussian noise obeying Eq. (4.20), i.e., noise whose intensity R vanishes, does not display the expected diffusive behavior. The random driving force has $R=0$, if contributions from regions of positive and negative correlations in the noise $f(t)$ cancel each other out. This counterbalance of positive correlations by negative ones leads to stochastic localization of free particles, a phenomenon first described for free particles driven by additive colored noise [23].

If Eq. (4.20) is not fulfilled, the stationary PDF does not exist. In this case, free particles display diffusive behavior that can be characterized by the fractional moments. The asymptotic behavior of the fractional moments is determined by the asymptotic behavior of $\sigma_q^2(t)$ as $t\to\infty$. Using Eqs. (4.5) and (4.16), we find for $m_r^0(t)$,

$$m_r^0(t)\sim\frac{\Gamma(\xi/2)}{\sqrt{2\pi}}2^{\xi/2}\sigma_q^{\xi-1}(t)\ (t\to\infty), \tag{4.23}$$

and using the asymptotic formula

$$D_{-\xi}(-z)-D_{-\xi}(z)\sim\frac{2^{(\xi+1)/2}}{\Gamma(\xi)}\Gamma\left(\frac{\xi+1}{2}\right)z, \tag{4.24}$$

$(z\to 0)$, which follows from Eq. (4.4), we obtain for $m_r^1(t)$,

$$m_r^1(t)\sim\frac{x_0^{1-\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{\xi+1}{2}\right)2^{\xi/2}\sigma_q^{\xi-2}(t)\ (t\to\infty). \tag{4.25}$$

If $0<R<\infty$, then $\sigma_q^2(t)\propto t$ as $t\to\infty$ [23], and Eqs. (4.23) and (4.25) yield $m_r^0(t)\propto t^{(\xi-1)/2}$ and $m_r^1(t)\propto t^{\xi/2-1}$. These relations show that $m_r^\nu(\infty)=0$ for $r\in(\alpha-1,\nu(1-\alpha))$, and $m_r^\nu(\infty)=\infty$ for $r\in(\nu(1-\alpha),\infty)$. Note that the dispersion of the particle position, $\sigma_x^2(t)=\langle x^2(t)\rangle-\langle x(t)\rangle^2$, can be represented as $\sigma_x^2(t)=m_2^0(t)-[m_1^1(t)]^2$. So, $\sigma_x^2(t)\sim m_2^0(t)\propto t^{1/(1-\alpha)}$ as $t\to\infty$, and the conditions $\alpha=0$, $\alpha<0$, and $0<\alpha<1$ correspond to normal diffusion, subdiffusion (diffusion slower than the normal), and superdiffusion (diffusion faster than the normal), respectively. In other words, in this case the state dependence of the noise (when $\alpha\neq 0$) gives rise to anomalous diffusive behavior.

For $R=\infty$, the function $\sigma_q^2(t)$ grows faster than t but slower than t^2 as $t\to\infty$ [23]. If $R(u)\propto u^{-\gamma}$ $(0<\gamma<1)$ as $u\to\infty$, then $\sigma_q^2(t)\propto t^{2-\gamma}$, and Eqs. (4.23) and (4.25) yield $m_r^0(t)\propto t^{(\xi-1)(1-\gamma/2)}$ and $m_r^1(t)\propto t^{(\xi-2)(1-\gamma/2)}$. Specifically, the long-time asymptotic behavior of the dispersion of the particle position has the form $\sigma_x^2(t)\propto t^{(2-\gamma)/(1-\alpha)}$. This result implies that normal diffusion, subdiffusion, and superdiffusion occur for $\alpha=\gamma-1$, $\alpha<\gamma-1$, and $\gamma-1<\alpha<1$, respectively. Note that there is a remarkable interrelation between $|x|^\alpha$-type multiplicative noises with finite and infinite intensities. Namely, multiplicative noise with infinite intensity ($R=\infty$) and characterized by the exponents $\alpha=\alpha'$ and γ leads to the same long-time asymptotic behavior of $m_r^0(t)$ [and $\sigma_x^2(t)$] as multiplicative noise with finite intensity $(0<R<\infty)$ and characterized by the exponent $\alpha=(1+\alpha'-\gamma)/(2-\gamma)$. In particular, the action of additive noise with $R=\infty$ is similar to the action of multiplicative noise with $0<R<\infty$ and $\alpha=(1-\gamma)/(2-\gamma)$.

3. $\alpha=1$, $\kappa>0$

If $R=0$, then the condition $\lim_{t\to\infty}\sigma_w^2(t)/t=0$ holds, and according to Eq. (4.9) all fractional moments $m_r(t)$ with $r>0$ tend to zero and all fractional moments with $r<0$ diverge as $t\to\infty$. Thus if the noise intensity R vanishes and if $\alpha=1$, the systematic force dominates the random force and drives the particle to the steady state, the minimum of the potential, $x=0$. The PDF approaches the Dirac delta function $\delta(x)$ as t goes to infinity. (The time evolution of the PDF is studied in more detail in the next section.) Note that this behavior is qualitatively different from the case of $\alpha<1$. In that case, the amplitude of the fluctuations does not go to zero linearly as $x\to 0$, and as discussed above, neither the systematic force nor the random force dominates in the long term; their effects balance and result in a stationary PDF.

If $R\neq 0$, then the long-time behavior is more complicated and no well-defined stationary PDF exists. This aspect will be addressed in more detail in the next section. As far as the fractional moments are concerned, we obtain the following results.

If $0<R<\infty$, then, writing the leading asymptotic term of $\sigma_w^2(t)$ as $2gt$, we obtain

$$m_r(t)\sim x_0^r\exp[r(rg-\kappa)t]\ (t\to\infty). \tag{4.26}$$

Thus $m_r(\infty)=\infty$ if $r<0$ or $r>\kappa/g$, and $m_r(\infty)=0$ if $0<r<\kappa/g$. For $r=\kappa/g$, the value $m_r(\infty)$ is determined by the second term of the asymptotic expansion of $\sigma_w^2(t)$. Finally, for $R=\infty$ the condition $\lim_{t\to\infty}\sigma_w^2(t)/t=\infty$ holds, and all fractional moments with $r\neq 0$ diverge as $t\to\infty$.

4. $\alpha=1$, $\kappa=0$

For the case of free particles, stochastic localization occurs again if Eq. (4.20) holds, since then we have $m_r(\infty)=x_0^r\exp[r^2\sigma_w^2(\infty)/2]<\infty$. According to Eq. (3.22) the stationary PDF exists in this case and has the form

$$P_{st}(x)=\frac{1}{\sqrt{2\pi}\sigma_w(\infty)x}\exp\left(-\frac{\ln^2(x/x_0)}{2\sigma_w^2(\infty)}\right). \tag{4.27}$$

Otherwise, $\sigma_w^2(\infty)=\infty$, and all fractional moments $m_r(t)=x_0^r\exp[r^2\sigma_w^2(t)/2]$ with $r\neq 0$ diverge as $t\to\infty$.

V. TIME EVOLUTION OF THE UNIVARIATE PDF

Having gained a first understanding of the temporal evolution of Eq. (1.1) by studying numerical characteristics of the PDF, namely, the fractional moments, we now investigate directly how the univariate PDF evolves with time. According to Eqs. (3.21), (3.22), and (4.10) the initial univariate PDF has the form $P_x(x,0)=\delta(x-x_0)$, which agrees with the initial condition $x(0)=x_0$ for Eq. (1.2). The temporal evolution of $P_x(x,t)$ depends on α, i.e., on the state dependence of the multiplicative noise and in particular on the strength of the random force near $x=0$. We first study the case $0<\alpha<1$. As discussed in the Introduction, the solution of Eq. (1.1) is not unique at $x=0$. We consider here the solution for which $x=0$ is a regular point. Nevertheless, both the systematic restoring force and the random driving force vanish at $x=0$. We, therefore, expect probability to accumulate in the neighborhood of this point. This is indeed the case. Equation (3.21) shows that for $t>0$ the PDF $P_x(x,t)$ has an absolute maximum $P_x(0,t)=\infty$ at $x=0$; $P_x(x,t)\sim|x|^{-\alpha}$ as $|x|\to 0$. The location of other extrema are given by the equation $\partial P_x(x,t)/\partial x=0$, which can be written in the form

$$|x|^{2(1-\alpha)}-x|x|^{-\alpha}x_0^{1-\alpha}e^{-\omega t}+\frac{\alpha}{1-\alpha}\sigma_q^2(t)=0. \quad (5.1)$$

This equation has solutions of the form

$$x_{\pm}(t)=x_0e^{-\kappa t}\left[\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{\alpha}{1-\alpha}\frac{1}{a^2(t)}}\right]^{1/(1-\alpha)}, \quad (5.2)$$

only if $a^2(t)\geqslant 4\alpha/(1-\alpha)$. For nondecreasing functions $\sigma_q^2(t)$, the condition $a^2(t)\geqslant 4\alpha/(1-\alpha)$ holds if $0\leqslant t\leqslant t_0$, where t_0 is the solution of the equation $a^2(t)=4\alpha/(1-\alpha)$. For $\kappa>0$ this equation always has a solution, and for $\kappa=0$ a solution exists if $\sigma_q^2(\infty)\geqslant x_0^{2(1-\alpha)}(1-\alpha)/4\alpha$. [If $\sigma_q^2(\infty)<x_0^{2(1-\alpha)}(1-\alpha)/4\alpha$, then Eq. (5.2) is valid for all t.] At $x=x_+(t)$ and $x=x_-(t)$, the PDF $P_x(x,t)$ has a local maximum and a local minimum, respectively. The former, $x_+(t)$, decreases monotonically with time, and the latter, $x_-(t)$, increases monotonically with time, if $t<t_0$. Specifically, in the short-time limit Eq. (5.2) yields

$$x_+(t)\sim x_0(1-\kappa t)\ \ (t\to 0), \quad (5.3)$$

for $\kappa>0$,

$$x_+(t)\sim x_0\left(1-\frac{\alpha}{(1-\alpha)^2}\frac{\sigma_q^2(t)}{x_0^{2(1-\alpha)}}\right)\ \ (t\to 0), \quad (5.4)$$

for $\kappa=0$, and

$$x_-(t)\sim\frac{1}{x_0}\left(\frac{\alpha}{1-\alpha}\sigma_q^2(t)\right)^{1/(1-\alpha)}\ \ (t\to 0), \quad (5.5)$$

for $\kappa\geqslant 0$, where $\sigma_q^2(t)$ is given by Eq. (4.10). At $t=t_0$, the local maximum and the local minimum coalesce, and for $t>t_0$ the univariate PDF has a single (infinite) maximum at $x=0$. If the equation $a^2(t)=4\alpha/(1-\alpha)$ has no solution, i.e., if t_0 does not exist, then the two local extrema of $P_x(x,t)$ exist for all times. The stronger the random driving force near zero relative to the systematic restoring force, i.e., the smaller α with $0<\alpha<1$, the longer the bimodality of the PDF lasts in time. Only for free particles, $\kappa=0$, i.e., only if the systematic restoring forces vanishes, can the bimodality persist forever. A necessary condition is the vanishing of the noise intensity, $R=0$, see Eq. (4.20).

To gain more insight into the behavior of $P_x(x,t)$ in the vicinity of $x=0$, we define the probability

$$W_\epsilon(t)=\int_{-\epsilon}^{\epsilon}dx P_x(x,t), \quad (5.6)$$

that $x(t)\in(-\epsilon,\epsilon)$. For the power-normal univariate PDF (3.21), Eq. (5.6) leads to the formula

$$W_\epsilon(t)=\frac{1}{2}\mathrm{erf}\left(\frac{a(t)}{\sqrt{2}}+\frac{\epsilon^{1-\alpha}}{\sqrt{2}\sigma_q(t)}\right)-\frac{1}{2}\mathrm{erf}\left(\frac{a(t)}{\sqrt{2}}-\frac{\epsilon^{1-\alpha}}{\sqrt{2}\sigma_q(t)}\right), \quad (5.7)$$

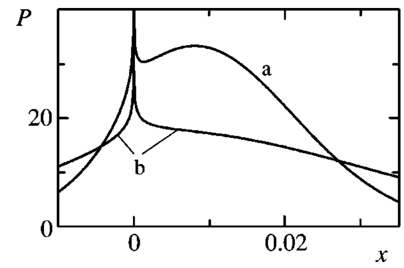

FIG. 1. Plot of the PDF $P_x(x,t)$ versus x for $\alpha=0.1$, $\kappa=0.1$, $x_0=0.01$. The correlation function $R(u)$ has the exponential form $R(u)=R(0)\exp(-u/t_c)$ with parameters $R(0)=0.01$ and $t_c=1$. The curves a and b correspond to $t=0.2$ and $t=0.4$, respectively.

which is valid for $\alpha<1$. Here $\mathrm{erf}(z)=1-\mathrm{erfc}(z)=(2/\sqrt{\pi})\int_0^z dt\exp(-t^2)$ is the error function. According to Eq. (5.7), we have $W_\epsilon(0)=0$ if $\epsilon<x_0$, $W_\epsilon(0)=1$ if $\epsilon>x_0$, $W_\epsilon(\infty)=\mathrm{erf}(\epsilon^{1-\alpha}/\sqrt{2}\sigma_q(\infty))$ if $\kappa>0$, $W_\epsilon(\infty)=0$ if $\kappa=0$ and $\sigma_q(\infty)=\infty$, and $W_\epsilon(t)\to 0$ as $\epsilon\to 0$. Specifically, the last condition shows that though $P_x(0,t)=\infty$, $x=0$ is indeed not an absorbing point.

To summarize, for $0<\alpha<1$ the PDF $P_x(x,t)$ evolves as follows. If $\kappa>0$ and $0<t<t_0$, then $P_x(x,t)$ is bimodal (see Fig. 1, curve a). With time, $x_+(t)$ and $P_x(x_+(t),t)$ decrease, $x_-(t)$ and $P_x(x_-(t),t)$ increase, and at $t=t_0$ the local extrema coalesce. For $t>t_0$, the PDF $P_x(x,t)$ is unimodal (see Fig. 1, curve b), and in the large-time limit it approaches the stationary distribution (4.18). If $\kappa=0$, then the temporal behavior of $P_x(x,t)$ depends on the value of $\sigma_q(\infty)$. For $\sigma_q^2(\infty)<x_0^{2(1-\alpha)}(1-\alpha)/4\alpha$ and $t>0$, the univariate PDF is bimodal as shown in Fig. 1 (curve a), and $P_{st}(x)$ is given by Eq. (4.22). For $x_0^{2(1-\alpha)}(1-\alpha)/4\alpha\leqslant\sigma_q^2(\infty)<\infty$ and $t<t_0$, the PDF $P_x(x,t)$ is bimodal as shown in Fig. 1 (curve a), whereas for $t\geqslant t_0$ it is unimodal as shown in Fig. 1 (curve b), and $P_{st}(x)$ is given again by Eq. (4.22). Finally, for $\sigma_q(\infty)=\infty$ the PDF $P_x(x,t)$ is bimodal for $t<t_0$ and unimodal for $t\geqslant t_0$, but the stationary PDF does not exist and $W_\epsilon(\infty)=0$ for any ϵ.

We now consider the case where $\alpha<0$, i.e., the amplitude of the multiplicative noise diverges as the particle approaches the minimum of the potential well. This is a useful model for exploring situations where the fluctuations drive the system out of the deterministic steady state, whereas the systematic force pushes the system towards it. According to Eq. (3.21), the univariate PDF $P_x(x,t)$ has an absolute minimum $P_x(0,t)=0$ at $x=0$, $P_x(x,t)\sim|x|^{|\alpha|}$ as $|x|\to 0$, and Eq. (5.1) has the solutions

$$x^{\pm}(t)=\pm x_0e^{-\kappa t}\left[\pm\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\alpha}{1-\alpha}\frac{1}{a^2(t)}}\right]^{1/(1-\alpha)}. \quad (5.8)$$

At $x=x^+(t)$, the PDF $P_x(x,t)$ has an absolute maximum, $P_x(x^+(t),t)$, and at $x=x^-(t)$, it has a local maximum, $P_x(x^-(t),t)$; $P_x(x^+(t),t)>P_x(x^-(t),t)$ for $t\neq\infty$. Using Eq. (5.8), we obtain $x^+(t)\sim x_+(t)(t\to 0)$ for $\kappa>0$,

$$x^{+}(t)\sim x_0\left(1+\frac{|\alpha|}{(1-\alpha)^2}\,\frac{\sigma_q^2(t)}{x_0^{2(1-\alpha)}}\right)\ (t\to 0), \qquad (5.9)$$

for $\kappa=0$, and

$$x^{-}(t)\sim -\frac{1}{x_0}\left(\frac{|\alpha|}{1-\alpha}\,\sigma_q^2(t)\right)^{1/(1-\alpha)}\quad (t\to 0), \qquad (5.10)$$

for $\kappa\geqslant 0$, where $\sigma_q^2(t)$ is given by Eq. (4.10). In the long-time limit Eq. (5.8) yields

$$x^{+}(\infty)=|x^{-}(\infty)|=\left(\frac{|\alpha|}{1-\alpha}\,\sigma_q^2(\infty)\right)^{1/2(1-\alpha)}, \qquad (5.11)$$

for $\kappa>0$, where $\sigma_q^2(\infty)$ is defined by Eq. (4.15), and

$$x^{+}(t)\sim|x^{-}(t)|\sim\left(\frac{|\alpha|}{1-\alpha}\,\sigma_q^2(t)\right)^{1/2(1-\alpha)}\quad (t\to\infty), \qquad (5.12)$$

for $\kappa=0$ and $\sigma_q(\infty)=\infty$, where $\sigma_q^2(t)$ is defined by Eq. (4.19). If $\kappa=0$ and $\sigma_q(\infty)<\infty$, Eq. (5.8) yields $|x^{\pm}(\infty)|<\infty$ and $x^{+}(\infty)>|x^{-}(\infty)|$. If $\kappa>0$ or $\kappa=0$ and Eq. (4.20) holds, then $P_x(x,t)$ approaches the stationary PDF (4.18) or (4.22), respectively, in the long-time limit. If $\kappa=0$ and Eq. (4.20) does not hold, then $|x^{\pm}(t)|\to\infty$ as $t\to\infty$, the stationary PDF does not exist, and $W_\epsilon(\infty)=0$ for all ϵ.

In summary, for multiplicative colored Gaussian noise whose amplitude diverges as $x\to 0$, the random driving force dominates near $x=0$. It drives the particle away from this point, and the probability density vanishes there. The noise acts symmetrically with respect to $x=0$, which results in temporal bimodality. For $\kappa>0$, the bimodal behavior of the PDF is stabilized in the long-time limit by the opposing effect of the systematic force, and the system evolves towards a stationary PDF. If the systematic force vanishes, $\kappa=0$, and the noise intensity of the random force also vanishes, $R=0$, i.e., stochastic localization of free particles occurs, then the bimodal behavior is stabilized by the balance between regions of positive and negative correlations of the noise $f(t)$. Again, the system evolves towards a stationary PDF. If $\kappa=0$ and the noise intensity is nonzero, then the most probable location of free particles goes to plus or minus infinity as $t\to\infty$, and a stationary PDF does not exist.

To illustrate the behavior of the PDF as a function of x, we plot $P_x(x,t)$ versus x for different values of α and t in Figs. 2 and 3. The correlation function of the Gaussian noise is again exponential as in Fig. 1. In this case $R=R(0)t_c$, and the PDF $P_x(x,t)$ approaches the stationary PDF (4.18) as $t\to\infty$.

For the case of additive noise, $\alpha=0$, the univariate PDF is Gaussian according to Eq. (3.21). As before, in this case the function $P_x(x,t)$ evolves with time to the stationary PDF (4.18) if $\kappa>0$, and to the stationary PDF (4.22) if $\kappa=0$ and $\dot{F}_0(t)=o(1/t)$ as $t\to\infty$. Otherwise the stationary PDF does not exist.

Finally, we consider the case $\alpha=1$, where the behavior of the PDF near $x=0$ is quite irregular as we will show. According to Eq. (3.22), $P_x(0,t)=P_x(\infty,t)=0$, and $P_x(x,t)$ is unimodal for all times. The maximum is located at $x=x_m(t)$,

$$x_m(t)=x_0\exp[-\sigma_w^2(t)-\kappa t], \qquad (5.13)$$

and

$$P_x(x_m(t),t)=\frac{1}{\sqrt{2\pi}\,\sigma_w(t)x_0}\exp[\sigma_w^2(t)/2+\kappa t]. \qquad (5.14)$$

If $\kappa=0$ and $\sigma_w(\infty)<\infty$, then $x_m(\infty)\neq 0$, $P_x(x_m(\infty),\infty)<\infty$, all fractional moments (4.9) are finite at $t=\infty$, and $P_x(x,t)$ approaches the stationary PDF (4.27) as $t\to\infty$. In all other cases we have $x_m(t)\to 0$ and $P_x(x_m(t),t)\to\infty$ as $t\to\infty$. Since $P_x(0,t)=0$, the long-time behavior of $P_x(x,t)$ in a small vicinity of $x=0$ is extremely irregular in those cases. To characterize $P_x(x,t)$ near zero, we write the probability $W_\epsilon(t)$, using Eqs. (5.6) and (3.22), as

$$W_\epsilon(t)=\frac{1}{2}\,\mathrm{erfc}\left(-\frac{\ln(\epsilon/x_0)+\kappa t}{\sqrt{2}\sigma_w(t)}\right). \qquad (5.15)$$

We define

$$\psi=\lim_{t\to\infty}\frac{\ln(\epsilon/x_0)+\kappa t}{\sqrt{2}\sigma_w(t)}, \qquad (5.16)$$

and taking into account that $\sigma_w^2(t)$ grows slower than t^2, we obtain $\psi=\infty$ if $\kappa>0$, and $\psi=0$ if $\kappa=0$ and $R\neq 0$. Since erfc $(-\psi)=2$ in the first case, and erfc $(-\psi)=1$ in the second case, Eq. (5.15) yields $W_\epsilon(\infty)=1$ and $W_\epsilon(\infty)=1/2$, re-

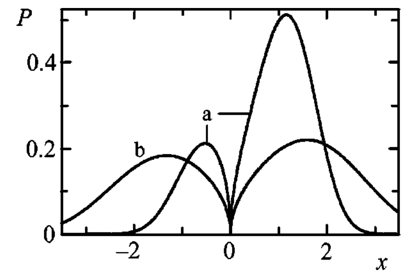

FIG. 2. Plot of the PDF $P_x(x,t)$ versus x for $\alpha=-0.5$, $\kappa=0.1$, $x_0=1$, $R(0)=1$, $t_c=1$, and $t=1$ (curve a), $t=5$ (curve b).

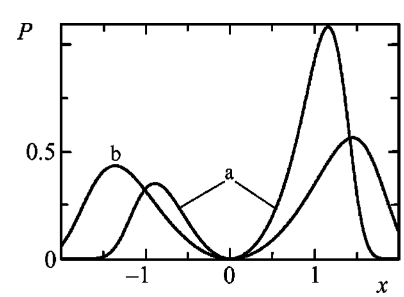

FIG. 3. Plot of the PDF $P_x(x,t)$ versus x for $\alpha=-2$, $\kappa=0.1$, $x_0=1$, $R(0)=1$, $t_c=1$, and $t=0.5$ (curve a), $t=2$ (curve b).

spectively. Though we have $x_m(\infty)=0$, $P_x(x_m(\infty),\infty)=\infty$, and $W_\epsilon(\infty)=1$ for $\kappa>0$, the limit $\lim_{t\to\infty} P_x(x,t)=\delta(x)$ holds only for $R=0$, when all fractional moments $m_r(\infty)$ with $r>0$ equal zero. If $R\neq 0$, i.e, if regions of positive correlations in the noise dominate, then the long-time behavior of the PDF is determined by the linear random driving force, no matter if a linear systematic restoring force exists, $\kappa>0$, or not, $\kappa=0$. The system will not approach a well-defined stationary PDF as $t\to\infty$. In other words, a linear systematic restoring force cannot balance the effects of linear multiplicative colored Gaussian noise, if $R\neq 0$. The amplitude of the noise must grow slower than $|x|$ as $|x|\to\infty$, if a stationary PDF is to exist for systems of type (1.1).

VI. COEFFICIENT OF CORRELATION

In the previous two sections we have characterized the temporal behavior of the Langevin equation (1.1) by studying the single-time PDF and its fractional moments. To obtain further insight into the effects of colored multiplicative Gaussian noise, we now turn our attention to a two-time quantity, the coefficient of correlation, in this section, and a pathwise quantity, the fractal dimension, in the next section. We define the coefficient of correlation of the random process $x(t)$ as usual by

$$r_x(t,t_1)=\frac{R_x(t,t_1)}{\sigma_x(t)\sigma_x(t_1)}, \tag{6.1}$$

where

$$R_x(t,t_1)=\langle x(t)x(t_1)\rangle-\langle x(t)\rangle\langle x(t_1)\rangle, \tag{6.2}$$

is the correlation function, and $\sigma_x^2(t)=R_x(t,t)$ is the dispersion of $x(t)$. Our focus here is the dependence of the limiting value $r_x(t,\infty)$ on the noise correlation function $R(u)$. First we consider the case of the power-normal distribution ($\alpha<1$). Using Eqs. (3.13) and (3.21) and a transformation of variables, we obtain

$$\begin{aligned}R_x(t,t_1)=&\frac{[\sigma_q(t)\sigma_q(t_1)]^{1/(1-\alpha)}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdy\,e^{-(x^2+y^2)/2}\\&\times[x+a(t)]|x+a(t)|^{\alpha/(1-\alpha)}\\&\times\{[\sqrt{1-r_q^2(t,t_1)}\,y+a(t_1)\\&+r_q(t,t_1)\,x]|\sqrt{1-r_q^2(t,t_1)}\,y+a(t_1)\\&+r_q(t,t_1)x|^{\alpha/(1-\alpha)}\\&-[y+a(t_1)]|y+a(t_1)|^{\alpha/(1-\alpha)}\}.\end{aligned} \tag{6.3}$$

Though the integral over y can be expressed by means of the Weber parabolic cylinder functions, we will use Eq. (6.3), which is more suitable for our purposes.

According to Eqs. (3.10) and (3.17), the coefficient of correlation $r_q(t,t_1)$ is given by

$$r_q(t,t_1)=\frac{e^{-\omega t_1}F_\omega(t)+e^{-\omega t}F_\omega(t_1)-F_\omega(t_1-t)}{2[e^{-\omega(t+t_1)}F_\omega(t)F_\omega(t_1)]^{1/2}}. \tag{6.4}$$

If $R(u)\sim c_\alpha u^{-\beta}$ $[0\leqslant\beta<1, c_\alpha=R(0)$ for $\beta=0]$ as $u\to 0$, then for $\omega>0$,

$$F_\omega(t)\sim\frac{c_\alpha}{(1-\beta)(2-\beta)}t^{2-\beta}\ (t\to 0), \tag{6.5}$$

which after some algebra leads to the expected result that $R_x(t,\infty)=0$ and $r_x(t,\infty)=0$, i.e., $x(t)$ and $x(\infty)$ are not correlated for an arbitrary correlation function $R(u)$ and $\kappa>0$.

For free particles, $\kappa=0$, i.e., $\omega=0$, stochastic localization can occur, and we expect the correlation between $x(t)$ and $x(t_1)$ to persist for $t_1\to\infty$. For $\kappa=0$, Eq. (6.4) is reduced to

$$r_q(t,t_1)=\frac{F_0(t)+F_0(t_1)-F_0(t_1-t)}{2[F_0(t)F_0(t_1)]^{1/2}}, \tag{6.6}$$

where according to Eq. (3.9) $F_0(z)$ is defined as

$$F_0(z)=\int_0^z duR(u)(z-u). \tag{6.7}$$

We use the relation

$$F_0(t_1)-F_0(t_1-t)=\int_{t_1-t}^{t_1}duR(u)(t_1-u)+t\int_0^{t_1-t}duR(u), \tag{6.8}$$

which follows from Eq. (6.7), the formula

$$\int_{t_1-t}^{t_1}duR(u)(t_1-u)\to 0\ (t_1\to\infty), \tag{6.9}$$

and the limit

$$\lim_{t_1\to\infty}\dot F_0(t_1)/\sqrt{F_0(t_1)}=0, \tag{6.10}$$

which for $0<R\leqslant\infty$ follows from the conditions $F_0(t_1)\sim t_1\dot F_0\ (t_1)(t_1\to\infty)$ and $\lim_{t_1\to\infty}\dot F_0(t_1)/t_1=0$, and for $R=0$ from $\dot F_0(\infty)=R$ and $F_0(\infty)>0$, to obtain for finite t,

$$r_q(t,\infty)=\frac{1}{2}\sqrt{F_0(t)/F_0(\infty)}. \tag{6.11}$$

According to this formula, $r_q(t,\infty)=0$ if $t=0$ or $F_0(\infty)=\infty$. The last condition holds if $0<R\leqslant\infty$ and also for $R=0$ if Eq. (4.20) does not hold. In contrast, if the condition (4.20) holds, i.e., stochastic localization of $x(t)$ occurs, then $F_0(\infty)<\infty, r_q(t,\infty)\neq 0$ and so $r_x(t,\infty)\neq 0$ for $t>0$. This result shows that correlations between $x(t)$ and $x(t+t_1)$ exist indeed even for $t_1\to\infty$ in the case of stochastic localization.

Next we consider the case of the lognormal distribution ($\alpha=1$). For this case we write the correlation function of $x(t)$ as

$$R_x(t,t_1)=\int_0^\infty\int_0^\infty dx\,dx_1 x x_1 P_x(x,t;x_1,t_1)-m_1(t)m_1(t_1). \tag{6.12}$$

Using Eq. (3.14) and introducing the new variables y and y_1

$$x=x_0 e^{y-\kappa t},\quad x_1=x_0 e^{y_1-\kappa t_1}, \tag{6.13}$$

we can reduce Eq. (6.12) to the form

$$R_x(t,t_1)=\frac{x_0^2 e^{-\kappa(t+t_1)}}{2\pi\sigma_w(t)\sigma_w(t_1)\sqrt{1-r_w^2(t,t_1)}}\times\int_{-\infty}^\infty\int_{-\infty}^\infty dy\,dy_1\exp\left[y+y_1-\frac{1}{2[1-r_w^2(t,t_1)]}\left(\frac{y^2}{\sigma_w^2(t)}+\frac{y_1^2}{\sigma_w^2(t_1)}-\frac{2r_w(t,t_1)}{\sigma_w(t)\sigma_w(t_1)}yy_1\right)\right]-m_1(t)m_1(t_1). \tag{6.14}$$

Performing the integration over y and y_1 in Eq. (6.14) and using Eq. (4.9) and the relation $\sigma_w^2(t)=2F_0(t)$, we obtain the explicit formulas for the correlation function

$$\begin{aligned}R_x(t,t_1)&=m_1(t)m_1(t_1)[e^{R_w(t,t_1)}-1]\\&=x_0^2\exp[F_0(t)+F_0(t_1)-\kappa(t_1+t)]\\&\quad\times\{\exp[F_0(t)+F_0(t_1)-F_0(t_1-t)]-1\},\end{aligned} \tag{6.15}$$

and for the coefficient of correlation

$$\begin{aligned}r_x(t,t_1)&=\frac{e^{R_w(t,t_1)}-1}{[(e^{R_w(t,t)}-1)(e^{R_w(t_1,t_1)}-1)]^{1/2}}\\&=\frac{\exp[F_0(t)+F_0(t_1)-F_0(t_1-t)]-1}{[(e^{2F_0(t)}-1)(e^{2F_0(t_1)}-1)]^{1/2}}.\end{aligned} \tag{6.16}$$

Specifically, if $t\to0$ and $R(u)\sim c_1u^{-\beta}$ $(0\leqslant\beta<1)$ as $u\to0$, then Eqs. (6.15) and (6.16) yield $R_x(t,t_1)\sim x_0m_1(t_1)\dot{F}_0(t_1)t$ and

$$r_x(t,t_1)\sim\sqrt{\frac{(1-\beta)(2-\beta)}{2c_1(e^{2F_0(t_1)}-1)}}\dot{F}_0(t_1)t^{\beta/2}, \tag{6.17}$$

$[c_1=R(0)$ for $\beta=0]$, i.e., $R_x(0,t_1)=0$ $(t_1>0)$ for all β, whereas $r_x(0,t_1)=0$ only for $0<\beta<1$. According to Eqs. (6.8)–(6.10), if $t_1=\infty$ then

$$r_x(t,\infty)=\frac{e^{F_0(t)}-1}{[(e^{2F_0(t)}-1)(e^{2F_0(\infty)}-1)]^{1/2}}, \tag{6.18}$$

if Eq. (4.20) holds, and $r_x(t,\infty)=0$ otherwise. Therefore, if stochastic localization of $x(t)$ occurs, then $r_x(t,\infty)\neq0$ $(t>0)$ for the power-normal as well as the lognormal distributions.

VII. FRACTAL DIMENSION

The fractal dimension d_f of a random process $x(t)$ characterizes the irregularity of $x(t)$ and can be defined in various ways [7,14]. Here we use the definition [34]

$$d_f=1+\lim_{\tau\to0}\frac{\ln\langle L_\tau\rangle}{\ln(1/\tau)}, \tag{7.1}$$

where

$$\langle L_\tau\rangle=\sum_{i=1}^N\langle\sqrt{(b\tau)^2+[x(t_i)-x(t_{i-1})]^2}\rangle, \tag{7.2}$$

is the average length of $x(t)$ on the interval $(t,t+\Delta t)$, $N\tau=\Delta t$, $t_i=t_{i-1}+\tau$, $t_0=t$, and b is a scaling parameter. In other words, d_f characterizes the fractal properties of $x(t)$ on the interval $(t,t+\Delta t)$. If this interval is small enough, so that the bivariate PDF of $x(t)$ does not change, then Eq. (7.2) is reduced to

$$\langle L_\tau\rangle=\frac{\Delta t}{\tau}\langle\sqrt{(b\tau)^2+[x(t+\tau)-x(t)]^2}\rangle. \tag{7.3}$$

Using Eq. (3.13), we can rewrite Eq. (7.3) for the power-normal distribution in the form

$$\begin{aligned}\langle L_\tau\rangle=\frac{\Delta t}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty dxdy\Bigg\{b^2+\frac{1}{\tau^2}\{&\sigma_q^{1/(1-\alpha)}(t+\tau)\\&\times[xr_q(t,t+\tau)+y\sqrt{1-r_q^2(t,t+\tau)}+a(t+\tau)]\\&\times|xr_q(t,t+\tau)+y\sqrt{1-r_q^2(t,t+\tau)}+a(t+\tau)|^{\alpha/(1-\alpha)}\\&-\sigma_q^{1/(1-\alpha)}(t)[x+a(t)]|x+a(t)|^{\alpha/(1-\alpha)}\}^2\Bigg\}^{1/2}\\&\times e^{-(x^2+y^2)/2}.\end{aligned} \tag{7.4}$$

If $R(u) \sim c_\alpha u^{-\beta}$ $(0 \leqslant \beta < 1)$ as $u \to 0$, then Eqs. (6.4) and (6.5) for $\tau \to 0$ yield $1 - r_q^2(t,t+\tau) \propto \tau^{2-\beta}$, and Eq. (7.4) leads to $\langle L_\tau \rangle \propto \tau^{-\beta/2}$. As a consequence, we obtain from Eq. (7.1)

$$d_f = 1 + \beta/2. \tag{7.5}$$

This result shows that the random processes with a power-normal distribution considered here have fractal properties only if $0 < \beta < 1$, i.e., only if the noise correlation function $R(u)$ has a singularity at $u = 0$. Note also that for such processes the fractal dimension d_f does not depend on t.

In the case of the lognormal distribution, Eq. (7.3) can be written as

$$\langle L_\tau \rangle = \frac{\Delta t}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \Bigg\{ b^2 + \frac{x_0^2}{\tau^2} \{ \exp[x \sigma_w(t) - \kappa t] - \exp[y \sigma_w(t+\tau) \sqrt{1 - r_w^2(t,t+\tau)} - \kappa(t+\tau) + x \sigma_w(t+\tau) r_w(t,t+\tau)] \}^2 \Bigg\}^{1/2} e^{-(x^2+y^2)/2}. \tag{7.6}$$

Since $1 - r_w^2(t,t+\tau) \propto \tau^{2-\beta}$ for $\tau \to 0$, Eq. (7.6) yields $\langle L_\tau \rangle \propto \tau^{-\beta/2}$, and the fractal dimension of random processes with a lognormal distribution is given by the same formula (7.5) as for the power-normal distribution.

VIII. CONCLUSIONS

We have studied the statistical properties of nonlinear systems driven by colored Gaussian noise whose amplitude depends on a power of the system state, $|x|^\alpha$. Starting from the exact solution $x(t)$ of the Langevin equation, we have obtained the univariate and bivariate PDFs of $x(t)$ and shown that, depending on the exponent α, the solution is described by the lognormal (if $\alpha = 1$) or the power-normal (if $\alpha < 1$) distribution. We have found that in both cases the system can exhibit the phenomenon of stochastic localization, i.e., a stationary univariate PDF for free particles exists, and we have derived the criterion when this occurs. We have studied in detail the time evolution of the univariate PDF, found exact expressions for the fractional moments of $x(t)$, and obtained and analyzed their short- and long-time asymptotics. Specifically, the long-time behavior of the dispersion of the particle position shows that diffusion of free particles can have anomalous character, and we have determined the conditions that lead to subdiffusion and superdiffusion.

Using the bivariate PDF, we have obtained an integral representation for the correlation function $R_x(t,t_1)$ and for the coefficient of correlation $r_x(t,t_1)$ of $x(t)$ for $\alpha < 1$, and for $\alpha = 1$ we have expressed $R_x(t,t_1)$ and $r_x(t,t_1)$ in terms of elementary functions. We have shown that if stochastic localization occurs, then $x(t)$ and $x(t+t_1)$ are correlated even as $t_1 \to \infty$, i.e., in the case of stochastic localization the condition $r_x(t,\infty) \neq 0$ $(t > 0)$ holds, and $r_x(t,\infty) = 0$ in all other cases. Also, we have calculated the fractal dimension d_f of $x(t)$ and established that $x(t)$ is fractal, i.e., $d_f > 1$, if the noise correlation function $R(u)$ has a power singularity at $u = 0$.

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